Defining functions

Functions can be treated as 'black boxes'. Basically, you put some numbers in, and you get an answer out, and to some extent it doesn't matter what happened in between. Here is an example of a function:

$$f(x) = x^2$$

Another way of expressing the same thing is:

$$f : x \to x^2$$

but this is less frequently seen.

Both of these mean that if you have some number, and do 'f' to it, then what you get out is the square of the number that you put in. For example,

$$f(3) = 9$$

because $3 \times 3 = 9$. Similarly, $f(7) = 49$ because $7 \times 7 = 49$. The variable x in the expression $f(x) = x^2$ is a dummy variable. Of itself, it has no meaning. It just means that when you work out $f(3)$ you substitute 3 for x in the expression, so $f(3) = 3^2 = 3 \times 3 = 9$.

This means that there is absolutely no difference between:

$$f(x) = x^2$$

and

$$f(y) = y^2.$$

They both define the same function, f. What the expressions concern is the function f, not the x or the y.

Here's another function:

$$g(x) = 3(x + 1)^2.$$

For this function, $g(3) = 48$ and $g(-1) = 0$.

Instead of putting a number into a function, we could put a letter in (letters stand for numbers anyway). So, for example,

$$g(p) = 3(p + 1)^2$$

Here the p has substituted for the x in the function definition. We could even put an algebraic expression into the function and see what we get out.

$$g(q + 3) = 3(q + 3 + 1)^2 = 3(q + 4)^2.$$

Even more ridiculous, we could put another function inside the function to see what we get out. At the moment, we have two functions,

$$f(x) = x^2 \quad (1)$$
$$g(x) = 3(x + 1)^2. \quad (2)$$

If we put $g(x)$ inside $f(x)$ we get:

$$f(g(x)) = f\left(3(x+1)^2\right) = \left(3(x+1)^2\right)^2 = 9(x+1)^4.$$

What is important to notice here, is that the inner function (in this case, $g(x)$) gets 'done' to x first, and then the outer function (in this case, $f(x)$). Here is another example, with two different functions:

$$a(x) = x + 3 \quad (3)$$
$$b(x) = x^2. \quad (4)$$

In this case,

$$a(b(x)) = x^2 + 3$$

and

$$b(a(x)) = (x+3)^2.$$

The order really does matter, of course. Suppose we put a number into these two expressions, say, 5, then

$$a(b(5)) = 5^2 + 3 = 28$$

but

$$b(a(5)) = (5+3)^2 = 64$$

When we put a function 'inside' another function, we could say we have a function 'of a' function. This is entirely different from getting two functions and multiplying them together. In the case of the functions a and b above, the product of the functions is:

$$a(x)b(x) = (x+3)x^2 = x^3 + 3x^2.$$

This is nothing like either $a(b(x))$ or $b(a(x))$. Taking the function of a function always means getting the output of one function and sticking it in as the input to another function.

There is a difference in notation, as well.

Product of a and b	$a(x)b(x)$
Output of a fed into b	$b(a(x))$
Output of b fed into a	$a(b(x))$

You might notice that when two things are multiplied together, there is no reason to write a multiplication sign ($\times$) to signify the multiplication. If you want to write *something* to ensure that everyone knows that a multiplication is what you meant, write a dot ($\cdot$) in the middle of the line. So, $2 \cdot 3$ means two times three, that is, six. Writing multiplication in this way is preferable, because otherwise it is easy to confuse the multiplication symbol with the letter x or the greek letter χ (called 'chi', pronounced 'kie') in complicated algebraic expressions. For instance, it is much clearer to write $\chi \cdot x$ to mean the product of χ and x, then if you were to write $\chi \times x$.