The temperature, T, of a mass of water in a kettle decreases according to the equation

$$\frac{dT}{dt} = -r(T - T_a)$$

where T_a is the ambient temperature. You boil some water, and leave it for 5 minutes once it has boiled. After these 5 minutes, its temperature is $60°$. After a further 5 minutes, its temperature is $40°$. What is the ambient temperature?

- Before we go any further, let's solve the differential equation. In this differential equation, $T - T_a$ is the difference between the temperature of the water, T, and the ambient, T_a, so what the equation is saying is that the temperature of the water decreases at a rate proportional to the difference between its current temperature and the ambient.
- To solve this differential equation, we should get all the Ts on one side, and all the ts on the other. To do this we can divide by $(T - T_a)$ and multiply by dt to give:

$$\frac{1}{T - T_a} dT = -r dt$$

- Now we can put integral signs in front of both sides, to give:

$$\int \frac{1}{T - T_a} dT = -\int r dt$$

  and integrate. Notice that the integral on the left hand side is easy because the top half of the fraction is the derivative of the bottom half, and so the integral is simply the logarithm of the bottom half.
- We obtain:

$$\ln(T - T_a) = -rt + c$$

  which would be much more convenient if we got rid of the ln bit. The ln is already standing by itself on the left hand side, so we can take the exponential of both sides, and doing the usual rearrangement with the arbitrary constant c gives:

$$T - T_a = A \exp(-rt)$$

  which we can rearrange to give an expression for the temperature as a function of time,

$$T = T_a + A \exp(-rt).$$

- We can fit the initial conditions. When $t = 0$ the water is boiling, that is, $T = 100$ (all our units are going to be in degrees from now on. This gives:

$$100 = T_a + A \exp(-r \cdot 0)$$

  and remembering that $\exp(0) = 1$ this gives an expression for A

$$100 = T_a + A$$

  Unfortunately this expression for A is in terms of T_a, which is a bit of a pain.

- The question provides plenty of additional information, however. It also says that at $t = 5$, $T = 60$ and that at $t = 10$, $T = 40$ so we can substitute these into the solution to give (along with the initial condition):

$$\begin{aligned}
100 &= T_a + A \quad (1)\\
60 &= T_a + A\exp(-5r) \quad (2)\\
40 &= T_a + A\exp(-10r) \quad (3)
\end{aligned}$$

- We now have some simultaneous equations to solve. We could rearrange them all in terms of T_a, but it is a lot quicker if we just subtract equation 2 off equation 1, and also subtract equation 3 off equation 1, to give:

$$\begin{aligned}
40 &= A\big(1 - \exp(-5r)\big) \quad (4)\\
60 &= A\big(1 - \exp(-10r)\big) \quad (5)
\end{aligned}$$

- This is great! We can now divide equation 5 by equation 4 to get an equation that just involves r:

$$\frac{3}{2} = \frac{1 - \exp(-10r)}{1 - \exp(-5r)}$$

  and rearranging this gives:

$$3(1 - \exp(-5r)) = 2(1 - \exp(-10r))$$

  and therefore

$$2\exp(-10r) - 3\exp(-5r) + 1 = 0$$

  which looks like a quadratic. It doesn't just *look* like a quadratic, though, it really *is* a quadratic. If you say $x = \exp(-5r)$ then x^2 is $\exp(-10r)$ which gives:

$$2x^2 - 3x + 1 = 0$$

- Solving this by the usual means gives $x = \frac{3\pm\sqrt{9-8}}{4}$ and so we have two possible solutions for r, namely $r = 0$ or $r = 0.138629...$ Referring back to equations 4 and 5 you can see that the $r = 0$ solution is ridiculous (it would require A to be ∞), so we'll use the $r = 0.138629...$ solution.

- Rearranging equation 4 gives:

$$A = \frac{40}{1 - \exp(-5r)}$$

  and substituting our newly-found value for r into this gives a solution for A, $A = 80$.

- Finally referring back to equation 1 and substituting in our newly-found value for A we obtain

$$T_a = 20$$

  so the ambient temperature (which is what we were asked about in the first place) is 20°.