Water flows into a reservoir at rate (in litres per day) $2 \times 10^6 \cdot r \cdot t\exp(-2t)$ after a rain-storm, where t is time expressed in days, and r is the duration of the rain-storm, also expressed in days. How much water flows into the reservoir (in total) per hour of rain?

i. Find the derivative of $At\exp(-Bt) + C\exp(-Bt)$ and show that if $A = BC$ this derivative turns into a simple expression in terms of $\exp(-Bt)$.

ii. Hence find the integral: $\int t\exp(-2t)\ \mathrm{d}t$?

iii. Hence determine the amount of water entering the reservoir for a rain-fall of duration r. You might find it useful to know that as x approaches ∞, $x\exp(-nx)$ gets closer and closer to zero if n is positive. Thereby calculate the amount of water which flows into the reservoir per hour of rain.

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i. Find the derivative of $At\exp(-Bt) + C\exp(-Bt)$.

- The two halves of this are $At\exp(-Bt)$ and $C\exp(-Bt)$. The first half is a product, of At with $\exp(-Bt)$ so the derivative of this is $A\exp(-Bt) - AB\exp(-Bt)$.
- Putting the whole thing back together gives $A\exp(-Bt) - ABt\exp(-Bt) - BC\exp(-Bt)$. It is better to write this as $(A - BC)\exp(-Bt) - ABt\exp(-Bt)$.
- If $A = BC$ (see the question) then the first half of this disappears to give just $-ABt\exp(-Bt)$.

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ii. Find the integral $\int t\exp(-2t)\ \mathrm{d}t$

- So far, we know that the derivative of $At\exp(-Bt)+C\exp(-Bt)$ is $-ABt\exp(-Bt)$ if $A = BC$ (we might want to write this as $C = A/B$.
- Integration is the reverse of differention, so another way of saying the same thing is that the integral: $\int -ABt\exp(-Bt)\ dt$ is $At\exp(-Bt) + C\exp(-Bt) + c$ if $C = A/B$ (note that I've stuck a constant of integration, $+c$, on the end.
- We actually want the integral $\int t\exp(-2t)\ \mathrm{d}t$. Comparing this with $\int -ABt\exp(-Bt)\ dt$ we get that $B = 2$ and $A = -1/2$. This gives $C = -1/4$. Therefore the integral that we want is

$$\int t\exp(-2t)\ \mathrm{d}t = -\frac{1}{2}t\exp(-2t) - \frac{1}{4}\exp(-2t) + c$$

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iii. Hence determine the amount of water per hour of rain

- The rate of water flow is given by $2 \times 10^6 \cdot r \cdot t\exp(-2t)$. The total amount of water that flows as the result of any rain-storm is therefore:

$$\text{total water flow} = 2 \times 10^6 r \int_0^\infty t\exp(-2t)dt,$$

that is, all the water that flows between time 0 and time ∞.

- We already worked out the solution to the indefinite integral (above, in part ii), so now we simply substitute this in to give:

$$\text{total water flow} = 2 \times 10^6 r \left[-\frac{1}{2} t \exp(-2t) - \frac{1}{4} \exp(-2t) \right]_0^\infty$$

More precisely, we should say that what we want is the limit of this value as the upper bound on the integral gets closer and closer to infinity; in other words we would more properly write:

$$\text{total water flow} = \lim_{t_{max} \to \infty} 2 \times 10^6 r \left[-\frac{1}{2} t \exp(-2t) - \frac{1}{4} \exp(-2t) \right]_0^{t_{max}}$$

...which looks slightly on the scary side; fortunately, we aren't being quite so rigorous here.

- Expanding out the 'less rigorous' version, we obtain:

$$\begin{aligned} \text{total water flow} \quad = \quad & 2 \times 10^6 r \left(\left(-\frac{1}{2} \cdot \infty \cdot \exp(-2 \cdot \infty) - \frac{1}{4} \exp(-2 \cdot \infty) \right) - \right. \\ & \left. \left(-\frac{1}{2} \cdot 0 \cdot \exp(-2 \cdot 0) - \frac{1}{4} \exp(-2 \cdot 0) \right) \right) \end{aligned}$$

Now, a 'proper' mathematician would have a fit if he/she read this, because we haven't taken limits in the way we should, and you shouldn't really have ∞ in the middle of formulae like this. Then said proper mathematician would go away and work on the 'more rigorous' version, and when he/she had finished, they'd end up with the same as us, namely:

$$\text{total water flow} = \frac{2 \times 10^6 r}{4} = 5 \times 10^5 r$$

(most of the terms either have $\exp(-\infty)$ in, which is 0, or have $0\times$ in them, which is 0 too). The only tricky one is $\infty \cdot \exp(-2 \cdot \infty)$; the ∞ in the front of the expression might make you think this is a very big number; the $-\infty$ inside the exponential might make you think it is very small. But the question specifies that this expression actually is pretty much zero—you read that $x \exp(-nx)$ tends towards 0 as x tends towards infinity).

Slightly in defence of the 'proper' mathematician, we have been slightly fortunate that the things we are dealing with are well-behaved (in a mathematical sense) and there are some instances where we'd have tripped up. This isn't worth worrying about too much for our purposes, however.

- r is the number of days in a rain-storm. Therefore, per hour of rain, we will have $5 \times 10^5/24$ litres of water coming into the reservoir, which is about 21,000 litres.