Find the area enclosed by the curves $y = x^2$ and $y = x + 6$.

- The curves cross when $x^2 = x + 6$. If you were to draw the curves out, you'd notice that the first curve $y = x^2$ has a minimum at $x = 0, y = 0$ and goes off upwards in both directions. The other curve is a straight line which goes through $y = 6$.
- Let's find out the x coordinates where the curves cross. $x^2 = x+a$ so $x^2-x-6 = 0$. This is a standard quadratic, with $A = 1$, $B = -1$ and $C = -a$. The quadratic formula says that $x = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$. In this case we get $x = \frac{1\pm\sqrt{1+4\cdot 6}}{2}$.
- So the two x extremes are $x = -2$ and $x = 3$.
- We want to know the area between these curves. The upper curve is the straight line $y = x+6$. The area between this curve and the axis is

$$\int_{-2}^{3} (x + 6)dx$$

  between the two extreme x values that we are interested in.
- The indefinite integral is $x^2/2 + 6x + c$. If you differentiate $x^2/2 + 6x + c$ you end up with $x + 6$ as required (the indefinite integral is the thing that you would have to differentiate to get back to the thing you are trying to integrate).
- So the integral is given by:

$$\left[\frac{x^2}{2} + 6x\right]_{-2}^{3} = 22.5 + 10 = 32.5.$$

- Now for the other curve. This time the indefinite integral is just

$$\int x^2 dx = x^3/3 + c$$

- The integral we want is given by:

$$\left[\frac{x^3}{3}\right]_{-2}^{3} = 9 + 8/3 = 35/3.$$

- We want the area between the curves, which is the area under the upper curve (32.5) minus the area under the lower curve (35/3). Sticking this into a calculator gives: 20.8333 which is the value that we want.

Solve the differential equation $dy/dx = (x+3)y$ **given that** $dy/dx = 9$ **when** $x = 0$.

- The first job is to separate the variables. In this case this is easy: we simply divide both sides by y and multiply both sides by dx to give

$$\frac{dy}{y} = (x+3)dx.$$

- Now we can integrate both sides. The integral of $1/y$ with respect to y is $\ln(y)$ and the integral of $x+3$ with respect to x is $x^2/2 + 3x$. We need to include a single constant of integration, c, to give:

$$\ln(y) = \frac{x^2}{2} + 3x + c.$$

- We can now try to get y in terms of x. Taking the exponential of both sides gives:

$$y = \exp\left(\frac{x^2}{2} + 3x + c\right)$$

  which we can write as

$$y = (\exp c)\exp\left(\frac{x^2}{2} + 3x\right).$$

  As c is just a constant of integration, we can write $A = \exp(c)$ to give:

$$y = A\exp\left(\frac{x^2}{2} + 3x\right).$$

- Now we need to fit the initial conditions. The initial condition is expressed in terms of the derivative, so it would be a good idea to take the derivative of y to find out what dy/dx actually is. To take the derivative we apply the function of a function rule (the inner function being $f(x) = \frac{x^2}{2} + 3x$ and the outer function being $f(x) = \exp(x)$).

- This gives

$$\frac{dy}{dx} = A(x+3)\exp(...)$$

  in which the ... corresponds to all the gunk in the bracket that I can't be bothered to write out again.

- When $x = 0$, all the stuff inside the exp evaluates to 0 so the exponential itself is 1. Hence,

$$\frac{dy}{dx} = 3A$$

  and applying the initial conditions we learn that $3A = 9$ and therefore $A = 3$.

- The solution of the differential equation is therefore:

$$y = 3\exp\left(\frac{x^2}{2} + 3x\right).$$