EMB exam, January 2000

Answer all questions. The duration of this exam is 45 minutes. You are expected to spend approximately 15 minutes on questions 1 and 2, and the remaining 30 minutes on questions 3 and 4.

Question 1

Differentiate the following expressions with respect to x:

1. $3x^2 - 7$;
2. $x \ln x$;
3. $x^2 \sin(2x) + \exp(x)$;
4. $9 - \sqrt{3 + 2x}$;
5. $\sin(x)\cos(x)$.

Question 2

Perform the following integrations:

1. $\int x^4 + \sin(x) \mathrm{d}x$;
2. $\int 3 + \exp(2x) \mathrm{d}x$;
3. $\int 1/5x \mathrm{d}x$;
4. $\int 3x + \sqrt{x} \mathrm{d}x$;
5. $\int_3^6 x^3 \mathrm{d}x$.

Question 3

1. Find an expression for $\int x \mathrm{d}x$;
2. Hence find the value of the integral $\int_4^{10} x \mathrm{d}x$;
3. Draw a diagram showing the area to which this value corresponds;
4. Without using an expression for the area of a triangle, show with geometry that the value of integral corresponds to the area you have drawn;

5. Find an expression for $\int_9^{10} x^2 \mathrm{d}x$;

6. Draw a diagram showing the area to which this value corresponds;

7. On the diagram, identify the largest rectangular region which is entirely contained by the area under the curve of x^2 with x between 9 and 10. Similarly, identify the smallest triangle which must be added to this rectangle to entirely contain the area under the same curve in the same region. Hence deduce upper and lower bounds for the integral $\int_9^{10} x^2 \mathrm{d}x$ and show that the true value of the integral lies between these bounds;

8. Find narrower bounds for the integral by performing the above analysis in two bits, between $x = 9$ and $x = 9.5$, and $x = 9.5$ and $x = 10$ respectively.

Question 4

1. There are 5 red and 5 blue balls in a bag. If I draw three balls out at random without replacement, what is the probability that they are (a) all blue, (b) a mixture of colours, (c) more blue than red?

2. If I do the same experiment, replacing the ball each time, what are the corresponding probabilities?

3. What would be the mean number of red and blue balls drawn out of the bag (a) assuming balls were not replaced (b) assuming balls were replaced?

4. If I do the same experiment, what is the probability that the second ball drawn out of the bag is red, if (a) balls are replaced (b) balls are not replaced.

5. If instead of drawing three balls out of the bag and then stopping, I draw balls out of the bag until I obtain a blue ball, what is (a) the mean number of balls I draw out of the bag (assuming I do not replace the balls), (b) the probability of me drawing more than 4 balls out of the bag (assuming I do not replace the balls), (c) the probability of me drawing more than 4 balls out of the bag (assuming I do replace the balls)?

6. Explain what is meant by (a) binomial probability, (b) the mean of a distribution, (c) the standard deviation of a distribution, (d) the variance of a distribution. Give expressions for these quantities.